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Analytical approach to the time-dependent probability density function in tilted periodic potentials

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In this work we introduce a scheme for the calculation of an approximate closed expression for the time-dependent probability density function for overdamped particles in tilted periodic potentials. Our derivation is based on an ansatz for the solution of the corresponding Fokker-Planck equation and on a self-consistent cumulant calculation. The high accuracy of our expression for the time-dependent probability density function is exhibited by comparisons with Langevin dynamics simulations and exact analytic results for the drift and diffusion coefficients. Good agreement is found both, for large and intermediate times.

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I. INTRODUCTION

Overdamped particles in periodic potentials, driven out of thermodynamical equilibrium, model many different systems such as Josephson junctions, electrophoretic particle separation, intracellular transport, etc. [1,2]. Due to this fact, a considerable amount of work has been devoted to the transport properties of such a model [2]. One of the simplest cases arises by considering a "static tilt" as the driving force that takes the particles out from equilibrium. Besides its prominent role in a number of physical systems, this model has also been used to study theoretically the foundations of nonequilibrium statistical mechanics [3,4] as well as to test some fundamental nonequilibrium relationships [5–7] recently proposed such as fluctuation theorems [8] or the Jarzynski equality [9]. Two of the transport properties that have been extensively studied are the flux of particles (or current) and the effective diffusion coefficient. In the case of the static tilted potential, an exact analytic formula for the flux of particles has been known for a long time [10], for arbitrary values of the strength of the tilt and of the temperature. For the effective diffusion coefficient two exact analytic expressions were derived rather recently [11,12]. Though the probability density function (PDF) for the steady state can be written in terms of quadratures involving the tilted potential, to our knowledge, a closed analytic expression for the evolution of the PDF towards the steady state is not yet available.

In this work we devise an approximate solution for the time-dependent Fokker-Planck equation of an ensemble of noninteracting particles in a tilted periodic potential. From the ensuing time-dependent PDF we derive analytical expressions for the current and the effective diffusion coefficient, which we compare with the exact results mentioned above. The time-dependent PDF, including the transport coefficients, shows very good agreement with numerical Langevin simulations.

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II. MODEL

The Brownian motion of an overdamped particle in a one-dimensional (1D) tilted periodic potential is modeled by the Langevin equation

$$\gamma \dot{x}(t) = f(x) + \xi(t), \tag{1}$$

where γ is the friction coefficient and $\xi(t)$ is a Gaussian white noise with zero mean and correlation $\langle \xi(t)\xi(t')\rangle = 2\gamma\beta^{-1}\delta(t-t')$. Here $f(x)=-V'(x)$ with $V(x)=V_p(x)-xF_0$ as the tilted potential, where $V_p(x)$ is the periodic part [with period L, i.e., $V_p(x+L)=V_p(x)$] and F_0 denotes the strength of the tilt. The friction coefficient γ will be set to 1 in the following. The Fokker-Planck equation that rules the dynamics of the PDF of an ensemble of noninteracting particles that follow the Langevin dynamics (1), is

$$\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial}{\partial x}[f(x)\rho(x,t)] + \frac{1}{\beta}\frac{\partial^2 \rho(x,t)}{\partial x^2}. \tag{2}$$

The time-dependent solution to (2) depends on the initial configuration of the ensemble $\rho(x,t=0)$ and on the boundary conditions. If $\rho(x,t)$ is a solution to (2) such that $\lim_{x\to\mp\infty}\rho(x,t)=0$ and $\int_{-\infty}^{\infty}\rho(x,t)dx=1$ for all t, we call it the extended PDF. On the other hand, when $\hat{\rho}(x,t)$ is a solution to (2) with boundary conditions $\hat{\rho}(x+L,t)=\hat{\rho}(x,t)$, it is called the reduced PDF. Both representations of the PDF are related through [2]

$$\hat{\rho}(x,t) = \sum_{n=-\infty}^{\infty} \rho(x+nL,t). \tag{3}$$

When $t\to\infty$ the reduced PDF tends to a finite limit $P_\infty(x)$ known as the steady-state PDF, which can be written as [2]

$$P_\infty(x) = \frac{1}{\mathcal{N}}\exp[-\beta V(x)]\int_x^{x+L}\exp[\beta V(y)]dy, \tag{4}$$

where $\mathcal{N}$ is a normalization constant given by

$$\mathcal{N}=\int_0^L \exp[-\beta V(x)]\int_x^{x+L}\exp[\beta V(y)]dydx. \tag{5}$$

Note that P_∞ is also the solution of the equation

$$f(x)P_\infty(x)-\frac{1}{\beta}\frac{dP_\infty}{dx}=J_\infty, \tag{6}$$

where J_∞ is the probability current of the system and it is given by the Stratonovich formula [10],

$$J_\infty=\frac{1-\exp(-\beta F_0 L)}{\beta\mathcal{N}}. \tag{7}$$

Furthermore, an exact expression for the diffusion coefficient of the problem is also known [12],

$$D_{ex}:=\lim_{t\to\infty}\frac{\langle x(t)^2\rangle-\langle x(t)\rangle^2}{t}=\frac{1}{\beta}\frac{\int_0^L\frac{dx}{L}I_-(x)I_+^2(x)}{\left(\int_0^L\frac{dx}{L}I_+(x)\right)^3}, \tag{8}$$

with $I_\pm(x)$ given by

$$I_\pm(x)=\int_0^L \exp[\pm\beta V_p(x)\mp\beta V_p(x\mp y)-yF_0]dy. \tag{9}$$

Additionally, for the time-dependent extended PDF, the Gaussian approximation,

$$\rho_g(x,t):=\frac{1}{N_g}\frac{P_\infty(x)}{\sqrt{4\pi D_{ex}t}}\exp\left(-\frac{(x-x_0-J_\infty Lt)^2}{4D_{ex}t}\right), \tag{10}$$

with N_g as a normalization factor, has been implemented in previous works [2,11].

III. TIME-DEPENDENT PROBABILITY DENSITY FUNCTION

In order to establish the main result of this paper we first introduce the function $S(x)$ as

$$S(x):=\frac{L}{\mathcal{M}}\int_0^x dx' e^{-\beta V_{\rm eff}(x')}\int_{x'}^{x'+L}dy e^{\beta V_{\rm eff}(y)}, \tag{11}$$

where $V_{\rm eff}$ is defined as

$$V_{\rm eff}(x):=V(x)+\frac{2}{\beta}\ln[P_\infty(x)], \tag{12}$$

and $\mathcal{M}$ is a normalization constant such that $S(L)=L$. We then adopt the following ansatz for the extended PDF $\rho(x,t)$ with initial condition $\rho(x,0)=\delta(x-x_0)$, which depends on two time-dependent quantities $\Pi(t)$ and $\Sigma(t)$,

$$\rho(x,t):=\frac{1}{N}\frac{P_\infty(x)}{\sqrt{2\pi\Sigma^2(t)}}\exp\left(-\frac{[S(x)-S(x_0)-\Pi(t)]^2}{2\Sigma^2(t)}\right), \tag{13}$$

with N a normalization constant. The above ansatz for the extended PDF guarantees a time-independent normalization for sufficiently large Σ [see Eq. (B10) in Appendix B]. The parameters $\Pi(t)$ and $\Sigma(t)$ are determined self-consistently as follows: First we calculate the first and the second cumulants of $\rho(x,t)$ in terms of Π and Σ^2. After that we calculate the time derivatives of the cumulants directly from the Fokker-Planck equation. Thereafter, we use the ansatz (13) in order to obtain $\dot{\Pi}$ and $\dot{\Sigma}^2$ also in terms of Π and Σ^2. This results in four expressions that involve four unknown quantities: The first two cumulants and the parameters Π and Σ^2. This set of differential equations can be reduced to two differential equations for the free parameters. The dynamics of Π and Σ^2 can be simplified if we work under the restriction $\Sigma^2\gg L^2$. Within this region, Π and Σ^2 coincide with the first and second cumulants, respectively, except for additive constants. Under these conditions they diverge linearly in time. Therefore, for the purposes of our calculations, we make the assumption that we can interchange the limit $t\to\infty$ with $\Sigma^2\to\infty$. Within this scheme we derive closed expressions for the current and for the effective diffusion coefficient.

A discussion on the origin of the function $S(x)$ is appropriate at this point. Our ansatz goes beyond the standard Gaussian approximation for the time-dependent PDF, which is obtained simply by replacing $S(x)$ by x, Π by $J_\infty Lt$ and Σ^2 by $2D_{ex}t$, where D_{ex} is the exact effective diffusion coefficient of Eq. (8). Such expression for the time-dependent PDF [2,11], with extended boundary conditions and initial condition $\rho(x,t)=\delta(x-x_0)$ is simpler than Eq. (13). However, when we attempt to solve Eq. (2) with the standard Gaussian approximation in the way described above (self-consistently), the resulting equations for the cumulants give a trivial effective diffusion coefficient, i.e., $\lim_{t\to\infty}\langle x(t)^2\rangle-\langle x(t)\rangle^2/t=\beta^{-1}$. In other words, the Gaussian approximation is unable to detect the periodic potential when we average the equation of motion of the second cumulant. This situation is avoided by introducing an inhomogeneous term within the exponential. This term is chosen from an analytic solution of (2) in the limit of strong tilt (see Appendix A). We can then perform a self-consistent calculation that gives an accurate description of the time dependence of the the PDF towards the nonequilibrium steady state for times large compared to the relaxation time. This procedure is supported by the fact that the transport coefficients obtained with this approach practically coincide with the exact formulas already calculated by alternative means, as we will see later on. Moreover, a recent WKB calculation of the decay rate for a specific tilted potential can be recovered with our approach since we are able to give general (approximate) expressions for the eigenvalues of the Fokker-Planck operator. One of the main assets of our treatment is that we can recover in a unified scheme several results obtained separately with different approaches by introducing only one self-consistent approximation.

We now proceed with the implementation of our calculations. We denote with $\langle x\rangle$ and $\sigma^2:=\langle x^2\rangle-\langle x\rangle^2$ the first and the second cumulants, respectively. In the $t\to\infty$ asymptotic limit one finds (see Appendixes B and C for detailed calculations),

$$\langle x\rangle=\Pi-\langle h(x)\rangle_\infty, \tag{14}$$

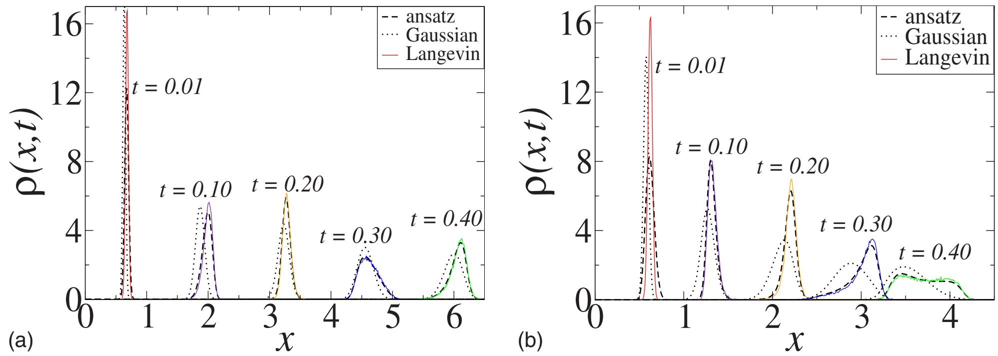

FIG. 1. (Color online) Time-dependent PDFs at β=50 determined at several times. (a) For F_0=15 and (b) for F_0=10. All the graphs display the three approaches used in this work, our ansatz of Eq. (13) (dashed line), the standard Gaussian approximation of Eq. (10) (dotted lines), and the Langevin simulation (solid lines). Notice the good agreement between the ansatz and the Langevin simulations.

$$\sigma^2 = \Sigma^2 + \langle h^2(x)\rangle_\infty - \langle h(x)\rangle_\infty^2, \tag{15}$$

where $h(x) \coloneqq S(x) - x$. Here $\langle \cdots \rangle_\infty$ stands for

$$\langle \cdots \rangle_\infty \coloneqq \int_0^L \cdots P_\infty(x)dx. \tag{16}$$

The time derivatives of $\langle x\rangle$ and σ^2 can be formally represented by integrals, after using the Fokker-Planck equation, as follows:

$$\frac{d\langle x\rangle}{dt} = \int_{-\infty}^{\infty} f(x)\rho(x,t)dx, \tag{17}$$

$$\frac{d\sigma^2}{dt} = \frac{2}{\beta} + 2\int_{-\infty}^{\infty} (x - \langle x\rangle)f(x)\rho(x,t)dx. \tag{18}$$

If we use the ansatz (13) within the last expressions we obtain, after some lengthy calculations (see Appendixes B and D), that

$$\frac{d\langle x\rangle}{dt} = \langle f(x)\rangle_\infty, \tag{19}$$

$$\frac{1}{2}\frac{d\sigma^2}{dt} = \frac{1}{\beta} + \langle h(x)\rangle_\infty\langle f(x)\rangle_\infty - \langle h(x)f(x)\rangle_\infty. \tag{20}$$

Note that the right-hand sides of expressions (19) and (20) are constant. This implies that $\langle x\rangle$ and σ^2 grow linearly in time. In fact Eqs. (19) and (20) are our prediction for the current

$$\langle v\rangle \coloneqq \lim_{t\to\infty}\langle \dot{x}\rangle = \langle f(x)\rangle_\infty = J_\infty L, \tag{21}$$

and the effective diffusion coefficient,

$$D_{\text{eff}} \coloneqq \lim_{t\to\infty}\frac{1}{2}\frac{d\sigma^2}{dt} = \frac{1}{\beta} + \langle h(x)\rangle_\infty\langle f(x)\rangle_\infty - \langle h(x)f(x)\rangle_\infty, \tag{22}$$

respectively. Moreover, from Eqs. (14) and (15) we see that the time dependence of the parameters Π and Σ^2 is approximately given by $\Pi = J_\infty t$, and $\Sigma^2 = 2D_{\text{eff}}t$, which, together with Eq. (13), gives us the explicit expression for time-dependent probability density function,

$$\rho(x,t) \coloneqq \frac{1}{N}\frac{P_\infty(x)}{\sqrt{4\pi D_{\text{eff}}t}}\exp\left(-\frac{[S(x) - S(x_0) - J_\infty Lt]^2}{4D_{\text{eff}}t}\right). \tag{23}$$

A. Strong tilting case

When F_0 is greater than the critical tilt, i.e., when $F_0 > 2\pi$, our self-consistent approach gives a good approximation to the time-dependent PDF because Eq. (23) stems from an asymptotic solution to the Fokker-Planck equation in the limit of strong tilt. We compare our approximate formula for the extended PDF with Langevin simulations and the standard Gaussian approximation of Eq. (10) in Figs. 1–3 for three different noise levels, β=50, β=10, and β=1, respectively. For simplicity we take L=1, and $V_p(x) = -\cos(2\pi x)$. Each figure exhibits the PDF for two different tilt strengths F_0=15 and F_0=10. All the particles are initially located at x_0=0.5, i.e., at the maximum of the periodic potential. At t=0 the PDF is then $\rho(x,0) = \delta(x - x_0)$. We see in Figs. 1 and 2 that our ansatz fits the Langevin simulations accurately, better than the standard Gaussian approximation for large tilt values.

It is worthwhile to point out that although our ansatz gives much better results than the Gaussian approximation for small and intermediate values of noise, in the presence of strong noise both calculations give good results as can be seen in Fig. 3. Actually our ansatz and the Gaussian approximation converge before they approach the Langevin simulations. Large noise implies that the particles spread rapidly through the potential as weakly perturbed random walkers; a scenario compatible with the Gaussian approximation. Despite the accuracy given by (10) for high temperature, when dealing with the effective diffusion coefficient the self-consistent calculation with (10) can only provide a rough estimate, namely, the noise strength β^{-1}, because at large temperature the noise fluctuations dominate over the deterministic part of the dynamics. Then, for a better description of the transport processes we require the information given by $S(x)$.

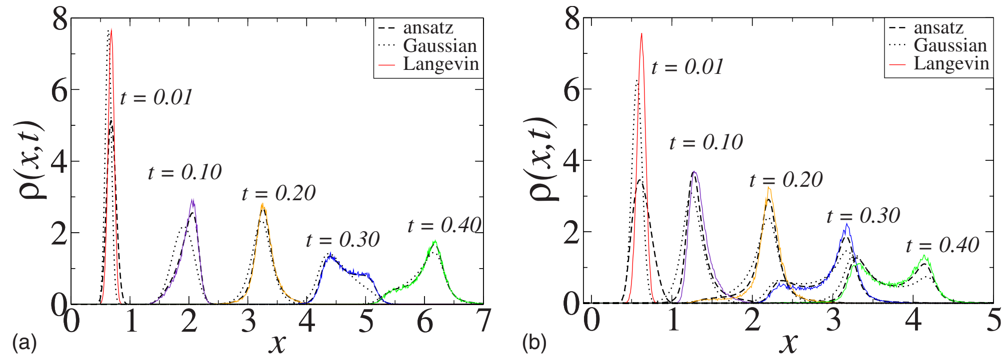

FIG. 2. (Color online) Time-dependent PDFs at $\beta=10$ determined at several times. (a) For $F_0=15$ and (b) for $F_0=10$. All the graphs display the three approaches used in this work, our ansatz of Eq. (13) (dashed line), the standard Gaussian approximation of Eq. (10) (dotted lines), and the Langevin simulation (solid lines). Notice the good agreement between the ansatz and the Langevin simulations.

Equation (23) is actually an approximate expression for the propagator of the Fokker-Planck equation. Thus, an extension of the time-dependent PDF to arbitrary initial conditions is straightforward. We write the propagator of Eq. (2) as

$$G(x,x_0,t) \coloneqq \frac{1}{N}\frac{P_\infty(x)}{\sqrt{4\pi D_{\mathrm{eff}}t}}e^{-[S(x)-S(x_0)-J_\infty Lt]^2/4D_{\mathrm{eff}}t}, \quad (24)$$

from which we can write the time dependence of the PDF with initial condition

$$\rho(x,0) = P_0(x)$$

as

$$\rho(x,t) = \int_{-\infty}^{\infty} P_0(x_0)G(x,x_0,t)dx_0. \quad (25)$$

Two examples showing the high accuracy of this expression for the time-dependent PDF at strong tilts are presented in Figs. 4 and 5. Figure 4 shows the evolution of the PDF at two different tilt strengths, $F_0=10$ and $F_0=15$, with a noise level $\beta^{-1}=10$ and an initial distribution P_0 given by

$$P_0(x) \coloneqq \begin{cases} 1/L_c & \text{if } |x-x_c| \leq L_c/2, \\ 0 & \text{if } |x-x_c| > L_c/2, \end{cases} \quad (26)$$

where L_c and x_c are two constants chosen as $L_c=1.3$ and $x_c=0.7$. We see that indeed, the ansatz for the time-dependent PDFs fits more accurately the Langevin simulations than the Gaussian approximation, which was determined using, instead of Eq. (24), the simpler propagator,

$$G_g(x,x_0,t) \coloneqq \frac{1}{N_g}\frac{P_\infty(x)}{\sqrt{4\pi D_{\mathrm{eff}}t}}e^{-(x-x_0-J_\infty Lt)^2/4D_{ex}t}. \quad (27)$$

The same behavior is observed if we use the initial distribution,

$$P_0(x) \coloneqq \begin{cases} \dfrac{1}{1+(x-x_c)^2} & \text{if } |x-x_c| \leq \tan(1/2), \\ 0 & \text{if } |x-x_c| > \tan(1/2), \end{cases} \quad (28)$$

where x_c is chosen again as $x_c=0.7$. The time evolution of the PDF with this initial distribution, compared with the ansatz and the Gaussian, is shown in Fig. 5. We can appreciate again that the better fits are obtained using our ansatz through Eqs. (24) and (25).

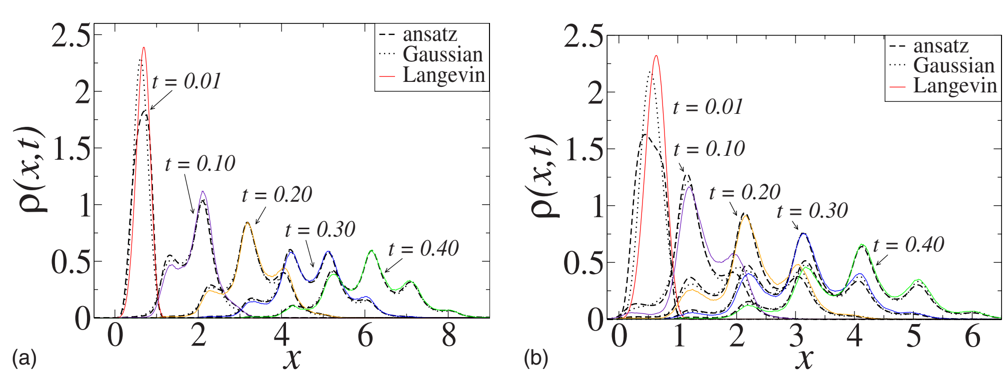

FIG. 3. (Color online) Time-dependent PDFs at $\beta=1$ determined at several times. (a) For $F_0=15$ and (b) for $F_0=10$. All the graphs display the three approaches used in this work, our ansatz of Eq. (13) (dashed line), the standard Gaussian approximation of Eq. (10) (dotted lines), and the Langevin simulation (solid lines). Notice the good agreement between the ansatz and the Langevin simulation for $F_0=15$ (a) and $F_0=10$ (b). A running average over 15 points has been done for each Langevin curve.

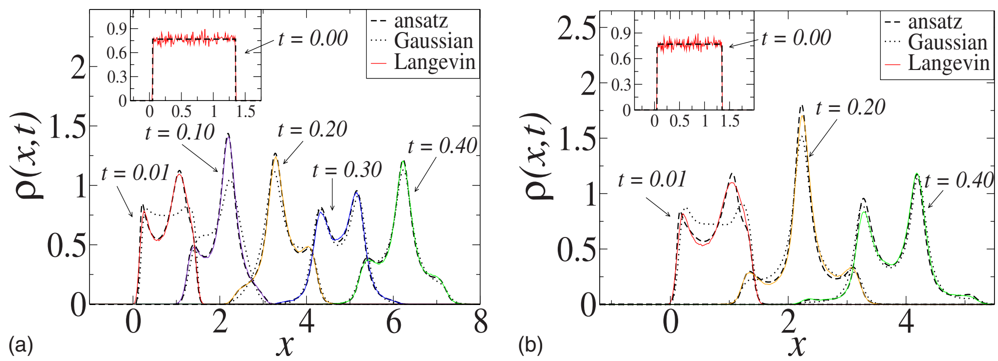

FIG. 4. (Color online) Time-dependent PDFs at $\beta=10$ with initial condition $\rho(x,0)=P_0(x)$. Here P_0 is the square distribution of Eq. (26) (see insets). (a) The PDFs for $F_0=15$ and (b) for $F_0=10$. Dashed lines indicate our ansatz, dotted lines indicate the Gaussian approximation, and solids lines are 15-point running averages of the Langevin calculation.

B. Small tilting case

At the critical tilt the ansatz for the PDF starts to exhibit deviations from Langevin simulations. This can be appreciated in Fig. 6. Notice that at intermediate times, the ansatz and the Gaussian approximation approach each other before reproducing the Langevin simulations accurately. This is a consequence of the fact that our calculation scheme ignores the transient dynamics of the cumulants towards the quasi-steady state.

At lower values of the tilt, the ansatz and the Gaussian approximation exhibit considerable deviations from the Langevin dynamics when all the particles are initially located near a local maximum of the tilted potential. This can be appreciated from Figs. 7(a) and 8(a). However if the particles start close to a minimum of the tilted potential, the ansatz fits the Langevin dynamics. This can be understood by the following arguments. The Langevin simulations of Fig. 7 manifest the presence of two time scales. The short time scale is a potential intrawell relaxation time, related to the displacement and widening of the initial δ distribution from the maximum to the minimum. The displacement is regulated by the slope of the potential and the widening depends on the curvature at the minimum. The longer time scale refers to the particle interwell transitions related to the potential amplitude through the Kramer's escape rate [14]. The PDF shift between the ansatz and the Langevin simulations observed when particles are placed initially at the potential maximum is present because at short times the cumulants acquire a transient time dependence from the intrawell relaxation processes. However, such relaxation processes are not taken into account in our main hypothesis that states that the cumulant dynamics is ruled by (19) and (20) only if the time is large. This shift is not observed when the system is initiated at the minima positions since there is no displacement due to the interwell relaxation.

C. Transport coefficients

Our closed expression (13) for ρ besides reproducing adequately the numerically calculated PDFs, especially for strong tilting, also allows the determination of the transport coefficients that compare remarkably well with previous exact calculations. The expression for the current we obtain reduces to the Stratonovich formula as we can see from Eqs. (6) and (19). Although our analytic expression for the effective diffusion coefficient differs from other known exact formulas [11,12], the numerical agreement is excellent. In fact, after evaluating the integrals involved in the calculation of βD_{eff}, both of our approximations and of Ref. [12], using the Simpson rule with an integration step of 10^{-4}, the observed average difference is below 10^{-3} for the parameters values used in Fig. 9. In the same figure numerical simulations of the system are also compared against our analytic expres-

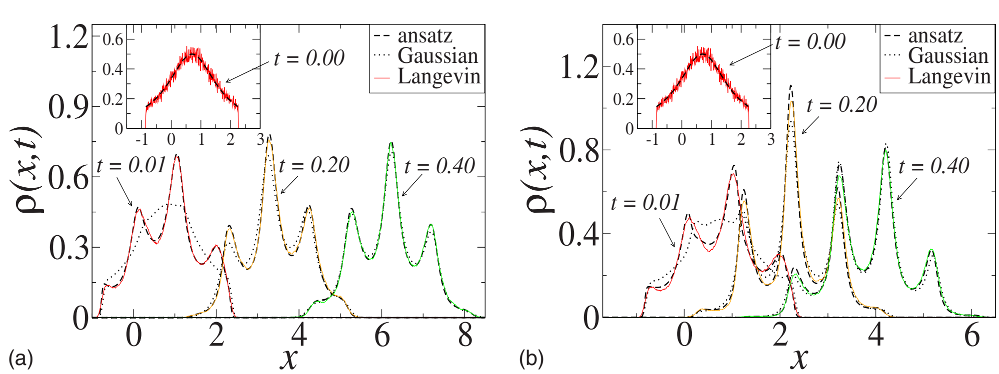

FIG. 5. (Color online) Time-dependent PDFs at $\beta=10$ with initial condition $\rho(x,0)=P_0(x)$. Here P_0 is the distribution given by Eq. (28) (see insets). (a) The PDFs for $F_0=15$ and (b) for $F_0=10$. Dashed lines indicate our ansatz, dotted lines indicate the Gaussian approximation, and solids lines are 15-point running averages of the Langevin calculation.

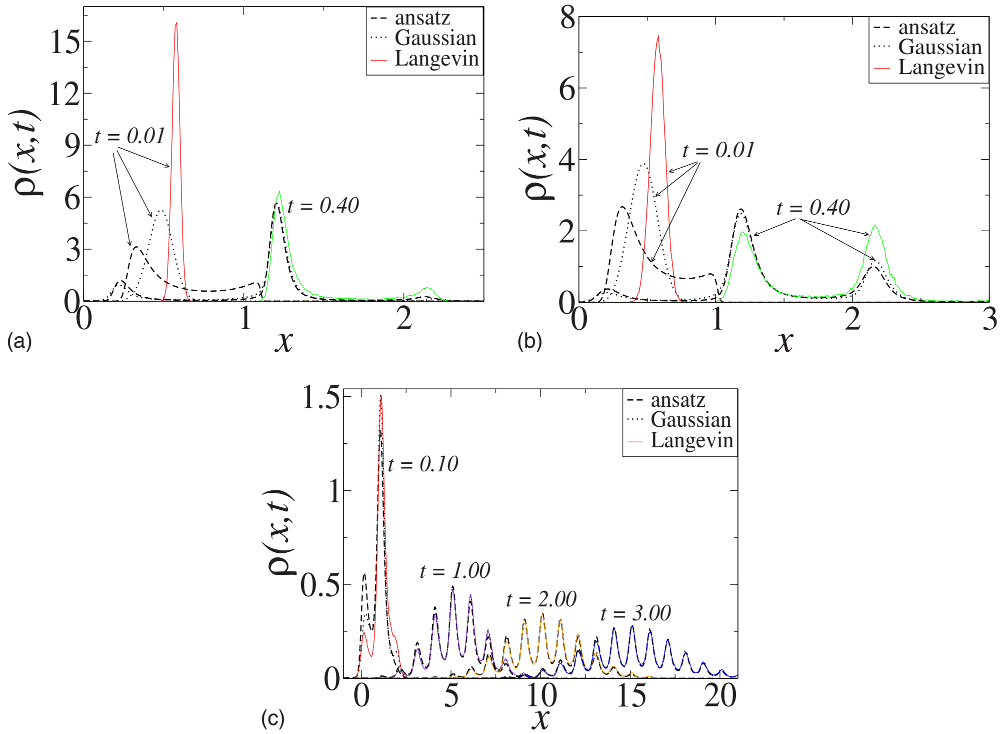

FIG. 6. (Color online) Time-dependent PDFs at the critical tilt $F_0=F_c=2\pi$ with initial condition $\rho(x,0)=\delta(x-x_0)$ at three different temperatures. Here $x_0=0.5$, which means that all the particles are initially located at a local maximum of the periodic potential. (a) The PDFs for $\beta=50$, (b) for $\beta=10$, and (c) for $\beta=1$. In the three noise regimes, both, the ansatz (dashed lines) and the Gaussian approximation (dotted lines) exhibit a considerable deviation from the numerical experiment (solid lines), except for longer times, as is shown in (c).

sion, D_{eff} versus F_0 is shown at two different temperatures, $\beta^{-1}=0.1$ and $\beta^{-1}=0.02$ for the tilted potential $V(x)=-\cos(2\pi x)-xF_0$.

Our analytic expression for D_{eff} also reproduces exactly various asymptotic limits. For example, in the case $F_0=0$, Eq. (20) reduces to the earlier result found for this quantity [13],

$$\beta D_{\text{eff}}=L^2/Z(\beta)Z(-\beta),$$

where $Z(\beta):=\int_0^L e^{-\beta V(x)}dx$, which can be verified by setting $F_0=0$ in our expression for D_{eff}.

Another asymptotic limit which is particularly important is $\beta\to\infty$ at the the critical tilt $F_0=F_c$, just where the potential loses all of its minima and gives rise to a third-order critical point. In this case a very singular phenomenon arises, originally reported by Reimann *et al.* [12]: Thermal diffusion undergoes a huge enhancement at low temperatures at the critical tilt. The dimensionless effective diffusion coefficient βD_{eff} diverges in the limit $\beta\to\infty$. Under the assumption that the potential V has only one third-order critical point per period when the tilt reaches its critical value $F_c>0$, the integrals involved in our expression for D_{eff} can be calculated

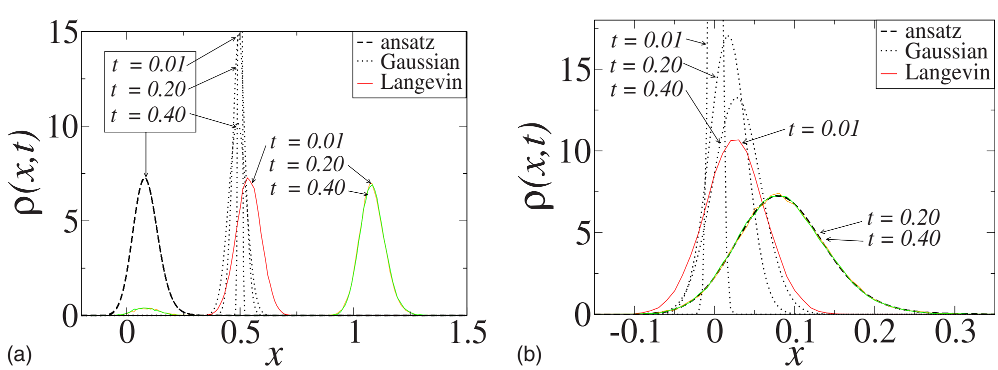

FIG. 7. (Color online) Time-dependent PDFs at $\beta=10$ for $F_0=3$, below the critical tilt. (a) The PDFs at three different times with initial condition $\rho(x,0)=\delta(x-0.5)$, i.e., all particles located at the maximum of the periodic potential at $t=0$. (b) The PDFs at three different times with initial condition $\rho(x,0)=\delta(x)$, i.e., with all particles located at the minimum of the periodic potential at $t=0$. It is interesting that good agreement between the ansatz (dashed lines) and the Langevin simulations (solid lines) is reached if the initial distribution is $\rho(x,0)=\delta(x)$, i.e., if all particles are initially located at a minimum of the periodic potential. We should also note that the our ansatz reaches rapidly the quasisteady state while the Gaussian approximation (dotted lines) does not.

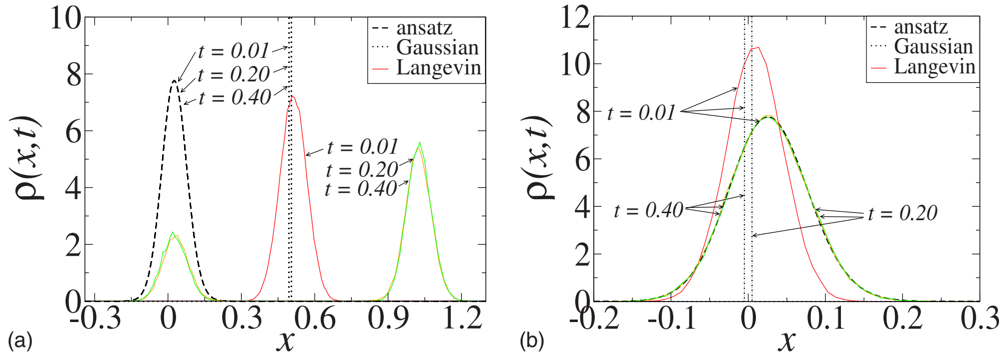

FIG. 8. (Color online) Time-dependent PDFs at $\beta=10$ for $F_0=1$, below the critical tilt. (a) The PDFs at three different times with initial condition $\rho(x,0)=\delta(x-0.5)$, i.e., all particles located at the maximum of the periodic potential at $t=0$. (b) The PDFs at three different times with initial condition $\rho(x,0)=\delta(x)$, i.e., with all particles located at the minimum of the periodic potential at $t=0$. It is interesting that good agreement between the ansatz (dashed lines) and the Langevin simulations (solid lines) is reached if the initial distribution is $\rho(x,0)=\delta(x)$, i.e., if all particles are initially located at a minimum of the periodic potential. We should also note that the our ansatz reaches rapidly the quasisteady state while the Gaussian approximation (dotted lines) does not.

approximately. In Appendix E we derive the following equation:

$$\beta D_{\text{eff}}=0.0699L^2(\beta\mu)^{2/3}, \tag{29}$$

where $\mu=-V'''(x_c)/3!$. This expression reproduces the divergence order of the exact result reported by Reimann *et al.* [12] for βD_{ex} with a relative difference in the prefactor of 4×10^{-3}. In Fig. 10 we plot the curve of βD_{eff} as a function of β and observe that our analytical expression fits accurately with the diffusion coefficient obtained from Langevin simulations, comparison with the exact result is also shown in the inset.

All these asymptotic expressions, as well as numerical evidence, raise the question as to whether Eq. (22) might even be an exact formula for the effective diffusion coefficient. So far we have been unable to prove the equivalence of Eq. (22) with the Reimann expression for D_{eff}.

It is also interesting to write the reduced PDF, using Eqs. (13) and (3) and Poisson's sum formula, as the series

$$\hat{\rho}(x,t)=P_\infty(x)\frac{1}{N}\sum_q e^{-iq\Pi-(1/2)q^2\Sigma^2}e^{iqS(x)}, \tag{30}$$

where $q=2\pi m/L$ and $m\in\mathbb{Z}$, since it can be interpreted as an approximate eigenfunction expansion of $\hat{\rho}$ [16]. From this result we note that the eigenvalues can be written in the form

$$\lambda_q=-iqJ_\infty-q^2D_{\text{eff}}, \tag{31}$$

which tells us that the relaxation towards the nonequilibrium steady state is ruled by the transport coefficients. Thus, Eq. (31) relates formally the decay rate (which is given by the

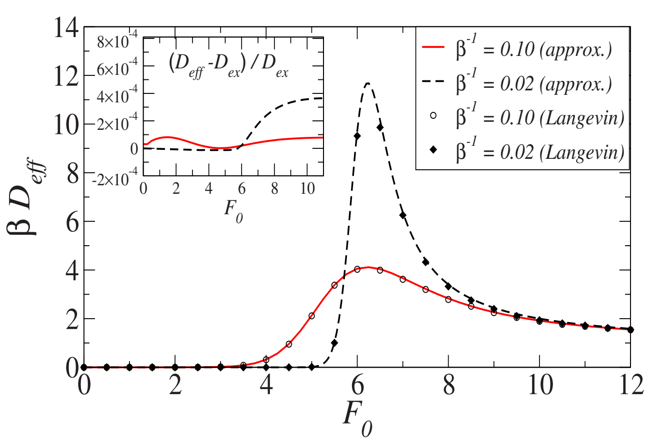

FIG. 9. (Color online) Effective diffusion coefficient for $V(x)=-\cos(2\pi x)-xF_0$. We show our analytical predictions for $\beta^{-1}=0.10$ (solid line) and $\beta^{-1}=0.02$ (dashed line). Langevin simulations are also plotted (open circles) for $\beta^{-1}=0.10$, and (filled diamonds) for $\beta^{-1}=0.02$. The calculation of D_{eff} from the Langevin simulations has an estimated error of 0.01. The curves for the relative effective diffusion coefficient calculated with the formula given in [12] overlap our analytic predictions since they differ in less than 10^{-3}. The inset shows the difference between these two calculations.

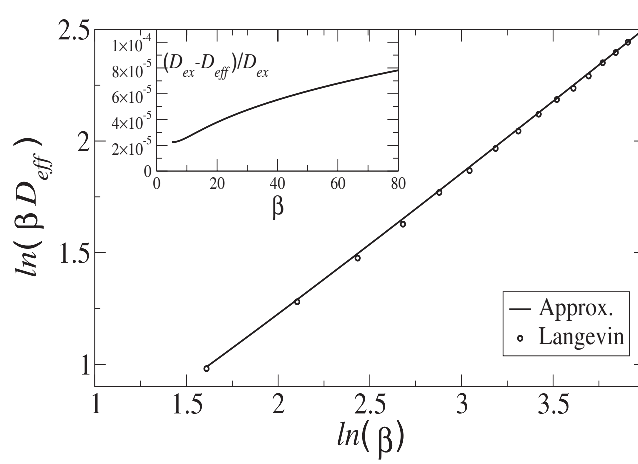

FIG. 10. Effective diffusion coefficient versus β at the critical tilt for a tilted cosine potential $V(x)=-\cos(2\pi x)-xF_0$. The log-log graph shows that βD_{eff} diverges with β as $\beta^{2/3}$. The solid line corresponds to our analytical expression and the open circles correspond to a Langevin dynamics simulation of 50 000 particles with a total simulation time of $T=100$. The inset shows the relative difference of our approximation with the exact result of Ref. [12].

real part of the first nonvanishing eigenvalue) to the effective diffusion coefficient. This means that the typical time scale for the particles to leave a local potential well should be given by $\tau = L/4\pi^2 D_{\text{eff}}$. The decay rate for a specific asymmetric tilted potential has been analyzed in Ref. [15] within the WKB approximation. In particular, Eq. (31) for $q=1$ reduces to the expressions found in Ref. [15] in the limit of large values of F_0 and β.

IV. CONCLUSIONS

Summing up, the assumption that ρ is given by Eq. (13) works very well once we calculate low order cumulants in a self-consistent way. The closed expression we obtain for the PDF fits accurately the PDFs obtained from the numerical integration of Langevin dynamics, not only in the long time limit, but also at intermediate times. Our studies also indicate that if the relaxation time increases, our approach fails to describe accurately the time-dependent behavior for short times. The latter is a consequence of our assumption on the asymptotic dynamics. We have also found that the standard Gaussian approximation results improve in the strong noise regime, the region where our ansatz tends rapidly to the Gaussian times the steady state, even before the latter approaches the Langevin simulations, in agreement with our assumptions. Transport coefficients determined with our ansatz show excellent agreement with previous exact calculations and Langevin simulations. Expression for the eigenvalues in terms of the drift an diffusion coefficient are given, thus relating the decay rate to the effective diffusion coefficient and giving the typical time that the particles spend in a local potential well before leaving it.

Finally, we would like to point out that the time-dependent tilting case, i.e., when the external forcing oscillates periodically in time, can be addressed with our formalism if the reduced PDF $P_\infty(x,t)$ is known exactly. Such an extension should enable the determination of a closed expression for the effective diffusion coefficient in terms of the asymptotic PDF, following the procedure outlined for the derivation of Eq. (20). However, this expression requires an analytical formula for the periodic solutions (in time and space) of the Fokker-Planck equation for the time-dependent-forcing case. This problem is the subject of current research.

ACKNOWLEDGMENTS

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APPENDIX A: ANALYTIC SOLUTION OF THE FOKKER-PLANCK EQUATION IN THE LIMIT OF STRONG TILT

Consider the Fokker-Planck equation

$$\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial}{\partial x}[f(x)\rho(x,t)] + \frac{1}{\beta}\frac{\partial^2 \rho(x,t)}{\partial x^2}, \tag{A1}$$

with $f(x)$ as a periodic function with period L, $f(x+L) = f(x)$. We thus make the separation ansatz that the time-dependent PDF can be written as the product of the steady-state PDF times a time-dependent function which we call $P(x,t)$, i.e., $\rho(x,t) = P_\infty(x)P(x,t)$. Substituting this into the above expression we obtain

$$\frac{\partial P}{\partial t} = -f_{\text{eff}}(x)\frac{\partial P}{\partial x} + \frac{1}{\beta}\frac{\partial^2 P}{\partial x^2}, \tag{A2}$$

where f_{eff} is defined as

$$f_{\text{eff}}(x) \coloneqq f(x) - \frac{2}{\beta}\frac{1}{P_\infty(x)}\frac{dP_\infty}{dx}. \tag{A3}$$

Now we assume that $P(x,t)$ depends on x through an invertible and continuous function $S(x)$ to be determined, i.e., we assume that P can be written as $P(x,t) = \phi[S(x),t]$. If we introduce this ansatz into Eq. (A2), we find that $\phi(u,t)$ satisfies the equation

$$\frac{\partial \varphi}{\partial t} = -K[\bar{S}(u)]\frac{\partial \varphi}{\partial u} + \frac{1}{\beta}\left(\frac{1}{\bar{S}'(u)}\right)^2 \frac{\partial^2 \varphi}{\partial^2 u}, \tag{A4}$$

where $\bar{S}(u)$ denotes the inverse of $S(x)$ and the prime denotes, as usual, the derivative of a function with respect to its argument. The function $K(x)$ inside the above equation is defined as

$$K(x) \coloneqq f_{\text{eff}}(x)S'(x) - \beta^{-1}S''(x). \tag{A5}$$

Notice that $S(x)$ is an arbitrary, invertible and continuous function. Once we fix $S(x)$, the function $K(x)$ is determined through Eq. (A5). The partial differential equation (A4) is then well-defined and a solution for $\phi(u,t)$ remains to be found. Among all the possibilities for $S(x)$ we choose the one that makes $K(x)$ a constant, which we will call $K_\infty L$. We thus obtain a differential equation for $S(x)$, namely,

$$\frac{d}{dx}[f_{\text{eff}}(x)S'(x)] - \beta^{-1}S'''(x) = 0. \tag{A6}$$

The above equation can be solved easily if we identify it with a stationary Fokker-Planck-like equation. Let Q_∞ be defined as $Q_\infty(x) \coloneqq LS'(x)$ and write Eq. (A6) in terms of Q_∞. We then obtain

$$\frac{d}{dx}[f_{\text{eff}}(x)Q_\infty(x)] - \frac{1}{\beta}\frac{d^2 Q_\infty}{dx^2} = 0, \tag{A7}$$

which is equivalent to

$$f_{\text{eff}}(x)Q_\infty(x) - \frac{1}{\beta}\frac{dQ_\infty}{dx} = K_\infty. \tag{A8}$$

Since $S(x)$ is arbitrary, we can solve Eq. (A8) with periodic boundary conditions. This choice makes $S(x)$ invertible in all $\mathbb{R}$. Then, the solution of Eq. (A8) can be written as

$$Q_\infty(x) = \frac{1}{\mathcal{M}} e^{-\beta V_{\text{eff}}(x)} \int_x^{x+L} dy e^{\beta V_{\text{eff}}(y)}, \tag{A9}$$

where it is convenient for our further calculations to choose the arbitrary constant $\mathcal{M}$ as a normalization factor, i.e.,

$$\mathcal{M} := \int_0^L dx e^{-\beta V_{\text{eff}}(x)} \int_x^{x+L} dy e^{\beta V_{\text{eff}}(y)}, \tag{A10}$$

and $V_{\text{eff}}(x)$ is defined as

$$V_{\text{eff}} := V(x) + \frac{2}{\beta}\ln(P_\infty).$$

Then the constant K_∞ can be written as

$$K_\infty = \frac{1 - e^{-\beta F_0 L}}{\beta \mathcal{M}},$$

and therefore the function $S(x)$ as

$$S(x) = L\int_0^x Q_\infty(x')dx'. \tag{A11}$$

With this choice of $S(x)$, Eq. (A4) becomes

$$\frac{\partial \varphi}{\partial t} = -K_\infty L \frac{\partial \varphi}{\partial u} + \frac{1}{\beta}\left(\frac{1}{\bar{S}'(u)}\right)^2 \frac{\partial^2 \varphi}{\partial u^2}. \tag{A12}$$

Notice that V_{eff} can be seen as an "effective tilted potential." The tilt of this potential is F_0 since $\ln[P_\infty(x)]$ is a periodic function. Moreover, by mere analogy with $P_\infty(x)$ in Eq. (6), we have that $Q_\infty(x)$ tends smoothly to $1/L$ as F_0 goes to infinity. This means that, in the large F_0 limit, $S'(x) \approx 1$ and therefore $S(x) \approx x$. Hence, the factor appearing in the second term of the last expression is approximately constant,

$$\left(\frac{1}{\bar{S}'(u)}\right)^2 = \{S'[\bar{S}(u)]\}^2 \approx 1, \tag{A13}$$

and then can be replaced by D_0,

$$D_0 = \frac{1}{\beta}\frac{1}{L}\int_0^L \left(\frac{1}{\bar{S}'(u)}\right)^2 dx.$$

We finally obtain that ϕ satisfies the equation

$$\frac{\partial \varphi}{\partial t} = -K_\infty L \frac{\partial \varphi}{\partial u} + D_0 \frac{\partial^2 \varphi}{\partial u^2}. \tag{A14}$$

whose solution is a Gaussian,

$$\varphi(u,t) = \frac{1}{\sqrt{4\pi D_0 t}} \exp\left(-\frac{(u - K_\infty L t)^2}{4D_0 t}\right).$$

It is important to remark that this approximation gives drift and diffusion coefficients K_∞ and D_0, respectively. It can be shown that K_∞ is actually the exact probability current, i.e., $K_\infty = J_\infty$. However, numerical evidence shows that D_0 gives a good description of the effective diffusion coefficient only for large values of F_0, i.e., above the critical tilt. This behavior is actually expected since we assumed F_0 large enough.

APPENDIX B: ASYMPTOTIC BEHAVIOR OF TIME-DEPENDENT AVERAGES

Here we show a result on the asymptotic behavior of averages with ansatz (13) as weight function. This lemma will be useful hereafter in the determination of the asymptotic behavior of the first two cumulants.

Let $\psi(u)$ be a periodic function, with period L. Its Fourier series is written as

$$\psi(u) = \sum_q \psi_q e^{iqu}, \tag{B1}$$

where $q := 2\pi n/L$ and $n \in \mathbb{Z}$.

Consider the integral

$$I := \frac{1}{\sqrt{2\pi\Sigma^2}} \int_{-\infty}^{\infty} \psi(u) e^{-(u-\Pi)^2/2\Sigma^2} du, \tag{B2}$$

then, substituting Eq. (B1) into Eq. (B2) we obtain,

$$I = \frac{1}{\sqrt{2\pi\Sigma^2}} \sum_q \psi_q \int_{-\infty}^{\infty} e^{iqu} e^{-(u-\Pi)^2/2\Sigma^2} du, \tag{B3}$$

which can be written as

$$I = \frac{1}{\sqrt{2\pi\Sigma^2}} \sum_q \psi_q e^{iq\Pi - \Sigma^2 q^2/2} \int_{-\infty}^{\infty} e^{-(u - \Pi - iq\Sigma^2)^2/2\Sigma^2} du, \tag{B4}$$

and then it follows that

$$I = \sum_q \psi_q e^{-\Sigma^2 q^2/2} e^{iq\Pi}. \tag{B5}$$

We should note that in the limit $\Sigma^2 \to \infty$ the integral I tends to the first term of the Fourier series of $\psi(u)$,

$$\lim_{\Sigma^2\to\infty} I = \frac{1}{L}\int_0^L \psi(u) du, \tag{B6}$$

which will be useful in the subsequent development of our approximation.

Now we are concerned with the asymptotic $\Sigma^2 \to \infty$ behavior of averages of functions depending on x weighted with the ansatz (13). Lets us first take the limit $\Sigma^2 \to \infty$ for the normalization constant N,

$$\lim_{\Sigma^2\to\infty} N = \lim_{\Sigma^2\to\infty} \frac{1}{\sqrt{2\pi\Sigma^2}} \int_{-\infty}^{\infty} P_\infty(x) e^{-[S(x)-\Pi]^2/2\Sigma^2} dx, \tag{B7}$$

which, after performing the change of variable $u = S(x)$, can be written as

$$\lim_{\Sigma^2\to\infty} N = \lim_{\Sigma^2\to\infty} \frac{1}{\sqrt{2\pi\Sigma^2}} \int_{-\infty}^{\infty} P_\infty[\bar{S}(u)]\bar{S}'(u) e^{-(u-\Pi)^2/2\Sigma^2} du, \tag{B8}$$

where $\bar{S}(u)$ denotes the inverse of $S(x)$ and the prime denotes, as usual, the derivative of a function with respect to its argument. Notice that $\bar{S}(u)$ is invertible by hypothesis and has the property $\bar{S}(u+L)=\bar{S}(u)+L$. Thus, $\bar{S}'(u)$ is a periodic function. Moreover, since $P_\infty[\bar{S}(u)]$ is a periodic function of u we can use the result (B6) to evaluate (B8) as follows:

$$\lim_{\Sigma^2\to\infty} N = \frac{1}{L}\int_0^L P_\infty[\bar{S}(u)]\bar{S}'(u)du, = \frac{1}{L}\int_0^L P_\infty(x)dx, \tag{B9}$$

therefore

$$\lim_{\Sigma^2\to\infty} N = \frac{1}{L}, \tag{B10}$$

which says that the normalization constant of the ansatz is well defined in the limit $\Sigma^2\to\infty$.

Now let $f(x)$ be an arbitrary periodic function with period L and consider the ensemble average $\langle f(x)\rangle$ using the ansatz for the probability density function of Eq. (13),

$$\langle f(x)\rangle \coloneqq \frac{1}{N\sqrt{2\pi\Sigma^2}}\int_{-\infty}^{\infty} f(x)P_\infty(x)e^{-[S(x)-\Pi]^2/2\Sigma^2}dx. \tag{B11}$$

Using the change of variable $u=S(x)$, we obtain

$$\langle f(x)\rangle = \frac{1}{N\sqrt{2\pi\Sigma^2}}\int_{-\infty}^{\infty} f[\bar{S}(u)]P_\infty[\bar{S}(u)]\bar{S}'(u)e^{-(u-\Pi)^2/2\Sigma^2}du. \tag{B12}$$

Notice that after the change of variable, the function inside the integral in (B12) can be seen as a periodic function times a Gaussian. This enables us to use expression (B5) in order to evaluate (B12) in terms of an infinite series. Let $\Phi(u)$ be a function defined as

$$\Phi(u) \coloneqq f[\bar{S}(u)]P_\infty[\bar{S}(u)]\bar{S}'(u), \tag{B13}$$

which can be expanded into a Fourier series with coefficients given by

$$\Phi_q = \frac{1}{L}\int_0^L f[\bar{S}(u)]P_\infty[\bar{S}(u)]\bar{S}'(u)e^{-iqu}du, \tag{B14}$$

or equivalently,

$$\Phi_q = \frac{1}{L}\int_0^L f(x)e^{-iqS(x)}P_\infty(x)dx, \tag{B15}$$

which we write as

$$\Phi_q = \frac{1}{L}\langle f(x)e^{-iqS(x)}\rangle_\infty, \tag{B16}$$

where $\langle\cdots\rangle_\infty$ means the stationary average, i.e., the average of a function weighted with the steady-state PDF,

$$\langle\cdots\rangle_\infty \coloneqq \int_0^L \cdots P_\infty(x)dx. \tag{B17}$$

Thus, using Eqs. (B12), (B2), and (B5),

$$\langle f(x)\rangle = \frac{1}{NL}\sum_q \langle f(x)e^{-iqS(x)}\rangle_\infty e^{iq\Pi-\Sigma^2q^2/2}, \tag{B18}$$

which is one of the main results of this appendix. In the limit $\Sigma^2\to\infty$ all of the terms in the series vanish with $e^{-\Sigma^2q^2/2}$ if $q\neq 0$. Therefore, only the first term in the series is finite in such a limit, and hence,

$$\lim_{\Sigma^2\to\infty}\langle f(x)\rangle \coloneqq \langle f(x)\rangle_\infty. \tag{B19}$$

This result states that time-dependent ensemble averages tend (exponentially fast) to the stationary average. In this sense, we say that the limits $\Sigma^2\to\infty$ and $t\to\infty$ are equivalent when we use our ansatz for the probability density function. An understanding of this behavior follows from Appendixes C and D where we show that Σ^2 is proportional to the second cumulant and diverges as time goes to infinity.

APPENDIX C: ASYMPTOTIC CUMULANTS

1. First cumulant

Here we calculate the first and the second cumulants of (13) in terms of the parameters Π and Σ^2. Let us start with $\langle x\rangle$ which can be written as

$$\langle x\rangle \coloneqq \frac{1}{N\sqrt{2\pi\Sigma^2}}\int_{-\infty}^{\infty} xP_\infty(x)e^{-[S(x)-\Pi]^2/2\Sigma^2}dx. \tag{C1}$$

Since $S(x)$ is invertible and has the property $S(x+L)=S(x)+L$, the function h defined as $h(x)\coloneqq S(x)-x$ is periodic, $h(x+L)=h(x)$, and has bounded amplitude $|h(x)-h(y)|\leq L$.

In order to calculate the asymptotic behavior of the first cumulant we write $\langle x\rangle$ as $\langle x\rangle=\langle S(x)\rangle-\langle[S(x)-x]\rangle=\langle S(x)\rangle-\langle h(x)\rangle$. The last term in this expression is the time-dependent average of a periodic function and therefore we can apply expression (B18) in order to evaluate it in terms of an infinite series. Nevertheless, let us start analyzing the time average of $S(x)$,

$$\begin{aligned}\langle S(x)\rangle &\coloneqq \frac{1}{N\sqrt{2\pi\Sigma^2}}\int_{-\infty}^{\infty} S(x)P_\infty(x)e^{-[S(x)-\Pi]^2/2\Sigma^2}dx\\ &= \frac{1}{N\sqrt{2\pi\Sigma^2}}\int_{-\infty}^{\infty} P_\infty[\bar{S}(u)]\bar{S}'(u)ue^{-(u-\Pi)^2/2\Sigma^2}du,\end{aligned} \tag{C2}$$

where we have used the change of variable $u=S(x)$. We write the factor $ue^{-(u-\Pi)^2/2\Sigma^2}$ as

$$ue^{-(u-\Pi)^2/2\Sigma^2} = -\Sigma^2\frac{d}{du}(e^{-(u-\Pi)^2/2\Sigma^2}) + \Pi e^{-(u-\Pi)^2/2\Sigma^2}, \tag{C3}$$

and we introduce it in Eq. (C2) to obtain

$$\langle S(x)\rangle = -\left(\frac{\Sigma^2}{N\sqrt{2\pi\Sigma^2}}\right)\int_{-\infty}^{\infty} P_\infty[\bar{S}(u)] \times \bar{S}'(u)\frac{d}{du}(e^{-(u-\Pi)^2/2\Sigma^2})du + \frac{\Pi}{N\sqrt{2\pi\Sigma^2}}\int_{-\infty}^{\infty} P_\infty[\bar{S}(u)]\bar{S}'(u)e^{-(u-\Pi)^2/2\Sigma^2}du,$$

which after a few calculations gives

$$\langle S(x)\rangle = \Pi + \frac{\Sigma^2}{N\sqrt{2\pi\Sigma^2}}\int_{-\infty}^{\infty}\frac{d}{du}\{P_\infty[\bar{S}(u)]\bar{S}'(u)\}e^{-(u-\Pi)^2/2\Sigma^2}du. \tag{C4}$$

Since the integral that appears in Eq. (C4) has the form (B2), we can evaluate it using expression (B5). To do this we define a periodic function $\psi(u)$ as

$$\psi(u) := \frac{d}{du}\{P_\infty[\bar{S}(u)]\bar{S}'(u)\}, \tag{C5}$$

which we can write as a Fourier series with coefficients given by

$$\psi_q = \frac{1}{L}\int_0^L \frac{d}{du}\{P_\infty[\bar{S}(u)]\bar{S}'(u)\}e^{-iqu}du = iq\frac{1}{L}\int_0^L P_\infty[\bar{S}(u)]\bar{S}'(u)e^{-iqu}du, \tag{C6}$$

expression that, after an appropriate change of variables, can be rewritten as $\psi_q = iq\langle e^{-iqS(x)}\rangle_\infty/L$, and hence Eq. (C4) takes the form

$$\langle S(x)\rangle = \Pi + \frac{\Sigma^2}{NL}\sum_q iq\langle e^{-iqS(x)}\rangle_\infty e^{iq\Pi - q^2\Sigma^2/2}. \tag{C7}$$

Recalling that $\langle x\rangle = \langle S(x)\rangle - \langle h(x)\rangle$ we can write the first cumulant as an infinite series using the result (C7) and expression (B18) to evaluate $\langle h(x)\rangle$. After this we obtain

$$\langle x\rangle = \Pi - \frac{1}{NL}\sum_q \langle [S(x)-x]e^{-iqS(x)}\rangle_\infty e^{iq\Pi - q^2\Sigma^2/2} + \frac{\Sigma^2}{NL}\sum_q iq\langle e^{-iqS(x)}\rangle_\infty e^{iq\Pi-q^2\Sigma^2/2}. \tag{C8}$$

The last result states that the first cumulant $\langle x\rangle$ has the asymptotic representation, $\langle x\rangle \approx \Pi - \langle [S(x)-x]\rangle_\infty$ in the limit $\Sigma^2 \to \infty$ which, as mentioned before, we consider as equivalent to the limit of large times. From this point of view, it will be useful to represent $\langle x\rangle$ as its asymptotic value plus a residual function $G_1(\Pi,\Sigma^2)$ that tends exponentially fast to zero with Σ^2,

$$\langle x\rangle = \Pi - \langle [S(x)-x]\rangle_\infty + G_1(\Pi,\Sigma^2), \tag{C9}$$

where,

$$G_1(\Pi,\Sigma^2) = \frac{\Sigma^2}{NL}\sum_q iq\langle e^{-iqS(x)}\rangle_\infty e^{iq\Pi-q^2\Sigma^2/2} - \frac{1}{NL}\sum_{q\neq 0}\langle [S(x)-x]e^{-iqS(x)}\rangle_\infty e^{iq\Pi-q^2\Sigma^2/2}, \tag{C10}$$

which goes to zero as $e^{-q^2\Sigma^2/2}$ since the term $q=0$ is absent in the series by definition.

2. Second cumulant

As in the case of the first cumulant, the second one σ^2, will be written in terms of $S(x)$ and $h(x)$. Straightforward computations give

$$\sigma^2 = \langle S^2(x)\rangle - \langle S(x)\rangle^2 + \langle h^2(x)\rangle - \langle h(x)\rangle^2 - 2[\langle S(x)h(x)\rangle - \langle S(x)\rangle\langle h(x)\rangle]. \tag{C11}$$

Some of the terms appearing in the above expression have been already calculated in the preceding section. Thus we only need to analyze the remaining ones.

Let us start with the average of S^2,

$$\langle S^2(x)\rangle := \frac{1}{N\sqrt{2\pi\Sigma^2}}\int_{-\infty}^{\infty} S^2(x)P_\infty(x)e^{-[S(x)-\Pi]^2/2\Sigma^2}dx = \frac{1}{N\sqrt{2\pi\Sigma^2}}\int_{-\infty}^{\infty} P_\infty[\bar{S}(u)]\bar{S}'(u)u^2 e^{-(u-\Pi)^2/2\Sigma^2}du, \tag{C12}$$

where we have used the change of variable $u=S(x)$. The exponential term can be written as

$$u^2 e^{-(u-\Pi)^2/2\Sigma^2} = \Sigma^4\frac{d^2}{du^2}(e^{-(u-\Pi)^2/2\Sigma^2}) - 2\Pi\Sigma^2\frac{d}{du}(e^{-(u-\Pi)^2/2\Sigma^2}) + (\Sigma^2+\Pi^2)e^{-(u-\Pi)^2/2\Sigma^2}, \tag{C13}$$

which can be incorporated in (C12), namely,

$$\langle S^2(x)\rangle = \frac{\Sigma^4}{N\sqrt{2\pi\Sigma^2}}\int_{-\infty}^{\infty} P_\infty[\bar{S}(u)]\bar{S}'(u)\frac{d^2}{du^2}(e^{-(u-\Pi)^2/2\Sigma^2})du - \frac{2\Pi\Sigma^2}{N\sqrt{2\pi\Sigma^2}}\int_{-\infty}^{\infty} P_\infty[\bar{S}(u)]\bar{S}'(u)\frac{d}{du}(e^{-(u-\Pi)^2/2\Sigma^2})du + (\Sigma^2+\Pi^2)\frac{1}{N\sqrt{2\pi\Sigma^2}} \times \int_{-\infty}^{\infty} P_\infty[\bar{S}(u)]\bar{S}'(u)e^{-(u-\Pi)^2/2\Sigma^2}du, \tag{C14}$$

further calculations give

$$\langle S^2(x)\rangle = \Sigma^2 + \Pi^2 + \frac{\Sigma^4}{N\sqrt{2\pi\Sigma^2}}\int_{-\infty}^{\infty}\frac{d^2}{du^2}\{P_\infty[\bar{S}(u)]\bar{S}'(u)\}e^{-(u-\Pi)^2/2\Sigma^2}du + \frac{2\Pi\Sigma^2}{N\sqrt{2\pi\Sigma^2}}\int_{-\infty}^{\infty}\frac{d}{du}\{P_\infty[\bar{S}(u)]\bar{S}'(u)\}e^{-(u-\Pi)^2/2\Sigma^2}du. \tag{C15}$$

Using expression (B5), the average of S^2 can be written as

$$\langle S^2(x)\rangle = \Sigma^2 + \Pi^2 - \frac{\Sigma^4}{NL}\sum_q q^2\langle e^{-iqS(x)}\rangle_\infty e^{iq\Pi - q^2\Sigma^2/2} + \frac{2\Pi\Sigma^2}{NL}\sum_q iq\langle e^{-iqS(x)}\rangle_\infty e^{iq\Pi - q^2\Sigma^2/2}. \tag{C16}$$

Now we calculate the average,

$$\langle S(x)h(x)\rangle = \frac{1}{N\sqrt{2\pi\Sigma^2}}\int_{-\infty}^{\infty}S(x)h(x)P_\infty(x)e^{-[S(x)-\Pi]^2/2\Sigma^2}dx,$$

using the change of variable $u=S(x)$ we have that

$$\langle S(x)h(x)\rangle = \frac{1}{N\sqrt{2\pi\Sigma^2}}\int_{-\infty}^{\infty}P_\infty[\bar{S}(u)]h[\bar{S}(u)]\bar{S}'(u)u \times e^{-(u-\Pi)^2/2\Sigma^2}du. \tag{C17}$$

Since $ue^{-(u-\Pi)^2/2\Sigma^2}$ can be written as in Eq. (C3), we obtain

$$\langle S(x)h(x)\rangle = -\frac{\Sigma^2}{N\sqrt{2\pi\Sigma^2}}\int_{-\infty}^{\infty}P_\infty[\bar{S}(u)]h[\bar{S}(u)]\bar{S}'(u)\frac{d}{du}\times(e^{-(u-\Pi)^2/2\Sigma^2})du + \frac{\Pi}{N\sqrt{2\pi\Sigma^2}}\int_{-\infty}^{\infty}P_\infty[\bar{S}(u)]h[\bar{S}(u)]\bar{S}'(u)\times e^{-(u-\Pi)^2/2\Sigma^2}du, \tag{C18}$$

which after a few manipulations and using the relation (B5) can be written as

$$\langle S(x)[S(x)-x]\rangle = \frac{\Sigma^2}{NL}\sum_q iq\langle h(x)e^{-iqS(x)}\rangle_\infty e^{iq\Pi - q^2\Sigma^2/2} + \frac{\Pi}{NL}\sum_q \langle h(x)e^{-iqS(x)}\rangle_\infty e^{iq\Pi - q^2\Sigma^2/2}. \tag{C19}$$

Finally, using expressions (C7), (C16), and (C19) and taking into account the relation (B18), we can write the second cumulant as follows:

$$\begin{aligned}\sigma^2 = \Sigma^2 &+ \frac{\Sigma^4}{NL}\sum_q (iq)^2\langle e^{-iqS(x)}\rangle_\infty e^{iq\Pi - q^2\Sigma^2/2} - \left(\frac{\Sigma^2}{NL}\sum_q iq\langle e^{-iqS(x)}\rangle_\infty e^{iq\Pi - q^2\Sigma^2/2}\right)^2 \\ &+ \frac{1}{NL}\sum_q \langle h^2(x)e^{-iqS(x)}\rangle_\infty e^{iq\Pi - q^2\Sigma^2/2} - \left(\frac{1}{NL}\sum_q \langle h(x)e^{-iqS(x)}\rangle_\infty e^{iq\Pi - q^2\Sigma^2/2}\right)^2 \\ &- 2\Bigg[\frac{\Sigma^2}{NL}\sum_q iq\langle h(x)e^{-iqS(x)}\rangle_\infty e^{iq\Pi - q^2\Sigma^2/2} - \left(\frac{\Sigma^2}{NL}\sum_q iq\langle e^{-iqS(x)}\rangle_\infty e^{iq\Pi - q^2\Sigma^2/2}\right) \\ &\times\left(\frac{1}{NL}\sum_q \langle h(x)e^{-iqS(x)}\rangle_\infty e^{iq\Pi - q^2\Sigma^2/2}\right)\Bigg],\end{aligned} \tag{C20}$$

with asymptotic behavior (in the limit $\Sigma^2\to\infty$) given by the $q=0$ term of the series, i.e., $\sigma^2\approx\Sigma^2+\langle h^2(x)\rangle_\infty-\langle h(x)\rangle_\infty^2$. We therefore write the second cumulant as its asymptotic value plus a function that goes to zero exponentially fast with Σ^2,

$$\sigma^2 = \Sigma^2 + \langle h^2(x)\rangle_\infty - \langle h(x)\rangle_\infty^2 + G_2(\Pi,\Sigma^2), \tag{C21}$$

where $G_2(\Pi,\Sigma^2)$ is defined as

$$\begin{aligned}G_2(\Pi,\Sigma^2) \coloneqq &\frac{\Sigma^4}{NL}\sum_q (iq)^2\langle e^{-iqS(x)}\rangle_\infty e^{iq\Pi - q^2\Sigma^2/2} - \left(\frac{\Sigma^2}{NL}\sum_q iq\langle e^{-iqS(x)}\rangle_\infty e^{iq\Pi - q^2\Sigma^2/2}\right)^2 \\ &+ \frac{1}{NL}\sum_{q\neq 0} \langle h^2(x)e^{-iqS(x)}\rangle_\infty e^{iq\Pi - q^2\Sigma^2/2} - \left(\frac{1}{NL}\sum_{q\neq 0} \langle h(x)e^{-iqS(x)}\rangle_\infty e^{iq\Pi - q^2\Sigma^2/2}\right)^2 \\ &- 2\Bigg[\frac{\Sigma^2}{NL}\sum_q iq\langle h(x)e^{-iqS(x)}\rangle_\infty e^{iq\Pi - q^2\Sigma^2/2} - \left(\frac{\Sigma^2}{NL}\sum_q iq\langle e^{-iqS(x)}\rangle_\infty e^{iq\Pi - q^2\Sigma^2/2}\right) \\ &\times\left(\frac{1}{NL}\sum_q \langle h(x)e^{-iqS(x)}\rangle_\infty e^{iq\Pi - q^2\Sigma^2/2}\right)\Bigg],\end{aligned} \tag{C22}$$

which is a series with terms that goes to zero either, as $e^{-q^2\Sigma^2/2}$, as $\Sigma^2 e^{-q^2\Sigma^2/2}$ or as $\Sigma^4 e^{-q^2\Sigma^2/2}$.

APPENDIX D: ASYMPTOTIC DYNAMICS

1. First cumulant

Here we calculate the asymptotic dynamics of the first and the second cumulants. We start evaluating the ensemble average of Eq. (17) in terms of the ansatz (13). This is easy to do since it is the ensemble average of a periodic function and we can use expression (B18), hence,

$$\frac{d\langle x\rangle}{dt}=\frac{1}{NL}\sum_q \langle f(x)e^{-iqS(x)}\rangle_\infty e^{iq\Pi-\Sigma^2q^2/2}, \tag{D1}$$

or writing separately the $q=0$ term from the series (which is its asymptotic value), we have

$$\frac{d\langle x\rangle}{dt}=\langle f(x)\rangle_\infty+F_1(\Pi,\Sigma^2), \tag{D2}$$

where F_1 is defined as

$$F_1(\Pi,\Sigma^2):=\frac{1}{NL}\sum_{q\neq 0}\langle f(x)e^{-iqS(x)}\rangle_\infty e^{iq\Pi-\Sigma^2q^2/2}. \tag{D3}$$

Equation (D2) shows that the particle current tends to $\langle f(x)\rangle_\infty$ in the limit $\Sigma^2\to\infty$, which leads to the well-known Stratonovich formula. This is not surprising since the ansatz for the PDF, in its reduced form, Eq. (30), tends to the stationary PDF,

$$\lim_{\Sigma^2\to\infty}\hat{\rho}(x,t)=P_\infty(x), \tag{D4}$$

because all of the terms in the series (30) vanish exponentially with Σ^2, except the $q=0$ element.

2. Second cumulant

Now we calculate the dynamics of the second cumulant. From Eq. (18) we have

$$\begin{aligned}\frac{d\sigma^2}{dt}&=\frac{2}{\beta}+2\int_{-\infty}^{\infty}(x-\langle x\rangle)f(x)\rho(x,t)dx\\&=\frac{2}{\beta}-\langle x\rangle\langle f(x)\rangle+2\int_{-\infty}^{\infty}xf(x)\rho(x,t)dx\\&=\frac{2}{\beta}-2\langle x\rangle\langle f(x)\rangle+2\int_{-\infty}^{\infty}S(x)f(x)\rho(x,t)dx\\&\quad-2\int_{-\infty}^{\infty}h(x)f(x)\rho(x,t)dx,\end{aligned} \tag{D5}$$

or in short-hand notation,

$$\frac{d\sigma^2}{dt}=\frac{2}{\beta}-2\langle x\rangle\langle f(x)\rangle-2\langle h(x)f(x)\rangle+2\langle S(x)f(x)\rangle. \tag{D6}$$

Recall that $\langle x\rangle$, $\langle f(x)\rangle$, and $\langle h(x)f(x)\rangle$ have already been calculated in the preceding sections. Thus, we only need to calculate $\langle S(x)f(x)\rangle$ using our ansatz,

$$\begin{aligned}\langle S(x)f(x)\rangle&=\frac{1}{N\sqrt{2\pi\Sigma^2}}\int_{-\infty}^{\infty}S(x)f(x)P_\infty(x)e^{-[S(x)-\Pi]^2/2\Sigma^2}dx\\&=\frac{1}{N\sqrt{2\pi\Sigma^2}}\int_{-\infty}^{\infty}f[\bar{S}(u)]P_\infty[\bar{S}(u)]\bar{S}'(u)u\\&\quad\times e^{-(u-\Pi)^2/2\Sigma^2}du,\end{aligned}$$

where we have used the change of variable $u=S(x)$. Now, using Eq. (C3) in the above expression we obtain

$$\begin{aligned}\langle S(x)f(x)\rangle&=-\frac{\Sigma^2}{N\sqrt{2\pi\Sigma^2}}\int_{-\infty}^{\infty}f[\bar{S}(u)]P_\infty[\bar{S}(u)]\bar{S}'(u)\\&\quad\times\frac{d}{du}(e^{-(u-\Pi)^2/2\Sigma^2})du\\&\quad+\frac{\Pi}{N\sqrt{2\pi\Sigma^2}}\int_{-\infty}^{\infty}f[\bar{S}(u)]P_\infty[\bar{S}(u)]\bar{S}'(u)\\&\quad\times e^{-(u-\Pi)^2/2\Sigma^2}du.\end{aligned} \tag{D7}$$

Performing some partial integrations and using expression (B5) we obtain

$$\begin{aligned}\langle S(x)f(x)\rangle&=\frac{\Sigma^2}{NL}\sum_q iq\langle f(x)e^{-iqS(x)}\rangle_\infty e^{iq\Pi-q^2\Sigma^2/2}\\&\quad+\frac{\Pi}{NL}\sum_q\langle f(x)e^{-iqS(x)}\rangle_\infty e^{iq\Pi-q^2\Sigma^2/2}.\end{aligned} \tag{D8}$$

The last result, together with Eqs. (D6), (C8), and (B5) leads to the following expression for the time derivative of the second cumulant:

$$\begin{aligned}\frac{1}{2}\frac{d\sigma^2}{dt}&=\frac{1}{\beta}-\frac{1}{NL}\sum_q\langle h(x)f(x)e^{-iqS(x)}\rangle_\infty e^{iq\Pi-q^2\Sigma^2/2}\\&\quad+\left(\frac{1}{NL}\sum_q\langle h(x)e^{-iqS(x)}\rangle_\infty e^{iq\Pi-q^2\Sigma^2/2}\right)\\&\quad\times\left(\frac{1}{NL}\sum_q\langle f(x)e^{-iqS(x)}\rangle_\infty e^{iq\Pi-q^2\Sigma^2/2}\right)\\&\quad-\left(\frac{\Sigma^2}{NL}\sum_q iq\langle e^{-iqS(x)}\rangle_\infty e^{iq\Pi-q^2\Sigma^2/2}\right)\\&\quad\times\left(\frac{1}{NL}\sum_q\langle f(x)e^{-iqS(x)}\rangle_\infty e^{iq\Pi-q^2\Sigma^2/2}\right)\\&\quad+\frac{\Sigma^2}{NL}\sum_q iq\langle f(x)e^{-iqS(x)}\rangle_\infty e^{iq\Pi-q^2\Sigma^2/2}.\end{aligned} \tag{D9}$$

Since the limit $\Sigma^2\to\infty$, the time derivative of the second cumulant reaches the value

$$\frac{1}{2}\frac{d\sigma^2}{dt}\approx\frac{1}{\beta}+\langle h(x)\rangle_\infty\langle f(x)\rangle_\infty-\langle h(x)f(x)\rangle_\infty, \tag{D10}$$

which comes from the $q=0$ terms of the series (D9), we can write

$$\frac{1}{2}\frac{d\sigma^2}{dt}=\frac{1}{\beta}+\langle h(x)\rangle_\infty\langle f(x)\rangle_\infty-\langle h(x)f(x)\rangle_\infty+F_2(\Pi,\Sigma^2), \tag{D11}$$

where

$$F_2(\Pi,\Sigma^2) = -\frac{1}{NL}\sum_{q\neq 0}\langle h(x)f(x)e^{-iqS(x)}\rangle_\infty e^{iq\Pi-q^2\Sigma^2/2} + \left(\frac{1}{NL}\sum_{q\neq 0}\langle h(x)e^{-iqS(x)}\rangle_\infty e^{iq\Pi-q^2\Sigma^2/2}\right)\times\left(\frac{1}{NL}\sum_{q\neq 0}\langle f(x)e^{-iqS(x)}\rangle_\infty e^{iq\Pi-q^2\Sigma^2/2}\right) - \left(\frac{\Sigma^2}{NL}\sum_{q} iq\langle e^{-iqS(x)}\rangle_\infty e^{iq\Pi-q^2\Sigma^2/2}\right)\times\left(\frac{1}{NL}\sum_{q}\langle f(x)e^{-iqS(x)}\rangle_\infty e^{iq\Pi-q^2\Sigma^2/2}\right) + \frac{\Sigma^2}{NL}\sum_{q} iq\langle f(x)e^{-iqS(x)}\rangle_\infty e^{iq\Pi-q^2\Sigma^2/2}, \quad \text{(D12)}$$

is a function that tends exponentially fast to zero as Σ^2 goes to infinity.

APPENDIX E: EFFECTIVE DIFFUSION COEFFICIENT IN THE LIMIT $\beta\to\infty$ AT THE CRITICAL TILT

The critical tilt F_c is defined as the forcing amplitude at which the potential wells disappear. Thus, whenever $F_0=F_c$, the tilted potential $V(x)$ has at least one saddle point x^* per period (one third-order critical point). We assume, as in Ref. [12], that there is only one saddle point per period and that the critical tilt is positive $F_c>0$. We will also assume that $x^*=0$, $V(x^*)=0$ without loss of generality. In this way we have that

$$V(x) \approx -\mu x^3, \quad \text{(E1)}$$

for x near the saddle point x^*. Here μ is a positive constant given by

$$\mu := -V'''(x^*)/3!. \quad \text{(E2)}$$

The first step to calculate the asymptotic behavior of the effective diffusion coefficient is to calculate an approximate expression for the stationary PDF,

$$P_\infty(x) = \frac{L}{\mathcal{N}}\exp[-\beta V(x)]\int_x^{x+L}\exp[\beta V(y)]dy. \quad \text{(E3)}$$

According to the steepest descent method, if $x\approx x^*$ the stationary PDF P_∞ can be approximated by

$$P_\infty(x) \approx \frac{1}{\mathcal{N}}e^{\beta\mu x^3}\int_x^\infty e^{-\beta\mu y^3}dy, \quad \text{(E4)}$$

where we have extended the upper integration limit up to infinity, which introduces an error that vanishes as $e^{-\beta F_0 L}$ as $\beta\to\infty$.

Analogously, the normalization constant $\mathcal{N}$ is given by

$$\mathcal{N} \approx \int_{-\infty}^\infty\int_0^\infty e^{\beta\mu[x^3-(x-y)^3]}dydx, \quad \text{(E5)}$$

since the maximum of the integrand

$$e^{-\beta V(x)}\int_x^{x+L}e^{\beta V(y)}dy, \quad \text{(E6)}$$

occurs at $x=x^*$. Hence we have

$$\mathcal{N} \approx (\mu\beta)^{-2/3}G_0, \quad \text{(E7)}$$

where G_0 is a constant given by

$$G_0 := \int_{-\infty}^\infty dz\int_0^\infty dt e^{z^3-(z+t)^3} = \frac{2^{1/3}\pi^{1/2}\Gamma\left(\frac{1}{6}\right)}{3^{3/2}}, \quad \text{(E8)}$$

and $\Gamma(z)$ is the Euler's γ function.

The expression (E4) for the stationary PDF can be written, as in Ref. [12], in terms of a dimensionless function defined as

$$K_0(z) := e^{z^3}\int_0^\infty e^{-(z+t)^3}dt, \quad \text{(E9)}$$

with which,

$$P_\infty(x) \approx \frac{(\mu\beta)^{1/3}}{G_0}K_0[(\mu\beta)^{1/3}x], \quad \text{(E10)}$$

where we have used Eq. (E7).

On the other hand, if x is not near x^*, we have that

$$\int_x^{x+L}e^{\beta V(y)}dy \approx e^{\beta V(x)}\int_x^{x+L}e^{\beta V'(x)(y-x)}dy, \quad \text{(E11)}$$

which, extending the upper integration limit to infinity and therefore introducing an error that vanishes as $e^{-\beta F_0 L}$, gives

$$\int_x^{x+L}e^{\beta V(y)}dy \approx \frac{e^{\beta V(x)}}{\beta f(x)}. \quad \text{(E12)}$$

This expression enables us to write the stationary PDF as

$$P_\infty(x) \approx \frac{\beta^{-1/3}\mu^{2/3}}{G_0 f(x)}. \quad \text{(E13)}$$

We also need to calculate an approximate expression for Q_∞,

$$Q_\infty(x) = \frac{1}{\mathcal{M}}e^{-\beta V_{\text{eff}}(x)}\int_x^{x+L}e^{\beta V_{\text{eff}}(y)}dy, \quad \text{(E14)}$$

where $V_{\text{eff}}(x)$ was defined as

$$V_{\text{eff}}(x) := V(x) + \frac{2}{\beta}\ln[P_\infty(x)]. \quad \text{(E15)}$$

A careful analysis shows that, at low temperatures, this "effective potential" V_{eff} has one maximum and one minimum near x^*. Indeed, in the limit $\beta\to\infty$, we have

$$\lim_{\beta\to\infty} V_{\text{eff}}(x) = V(x), \quad \text{(E16)}$$

which can be seen from Eqs. (E10), (E13), and (E15). This implies, according to the steepest descent method, that the main contribution to the integral involved in Eq. (E14) comes from points near x^*. Thus, we only need a good representation of the integrand in this zone. If we set the explicit form of V_{eff} into Eq. (E14) we obtain

$$Q_\infty(x) = \frac{1}{\mathcal{M}} \int_x^{x+L} \left(\frac{P_\infty(y)}{P_\infty(x)}\right)^2 e^{-\beta[V(x)-V(y)]} dy.$$

Now we expand the tilted potentials up to third order, $V(x) - V(y) \approx -\mu(x^3 - y^3)$, to get

$$Q_\infty(x) \approx \frac{1}{\mathcal{M}} \int_x^{x+L} \left(\frac{P_\infty(y)}{P_\infty(x)}\right)^2 e^{\beta\mu(x^3-y^3)} dy.$$

Since we are interested in an asymptotic representation of Q_∞ in the neighborhood of x^* we can write $P_\infty(x) = (\beta\mu)^{1/3} K_0[(\beta\mu)^{1/3}x]/G_0$ inside the above expression. Moreover, since the major contribution to the integral occurs for those values of y which are near x^*, we can also approximate $P_\infty(y)$ by $(\beta\mu)^{1/3}K_0[(\beta\mu)^{1/3}y]/G_0$. In this way we obtain

$$Q_\infty(x) \approx \frac{1}{\mathcal{M}} \int_x^{\infty} \left(\frac{K_0[(\beta\mu)^{1/3}y]}{K_0[(\beta\mu)^{1/3}x]}\right)^2 e^{\beta\mu(x^3-y^3)} dy,$$

where we have extended the upper integration limit to infinity introducing an error that goes to zero as $e^{-\beta F_c L}$. In order to simplify notation we introduce a dimensionless function as

$$K_1(z) := \frac{e^{z^3}}{K_0^2(z)} \int_0^\infty K_0^2(z+t) e^{-(z+t)^3} dt, \tag{E17}$$

and write

$$Q_\infty(x) \approx \frac{(\mu\beta)^{1/3}}{G_1} K_1[(\mu\beta)^{1/3}x], \tag{E18}$$

where G_1 is given by

$$G_1 := \int_{-\infty}^{\infty} dz \int_0^\infty dt \left(\frac{K_0(z+t)}{K_0(z)}\right)^2 e^{z^3-(z+t)^3}. \tag{E19}$$

Notice that, for $x \approx x^*$, the function $S(x)$ has the asymptotic representation,

$$S(x) \approx LT[(\mu\beta)^{1/3}x], \tag{E20}$$

where T is a dimensionless function given by

$$T(z) := \frac{1}{G_1} \int_0^z K_1(t) dt. \tag{E21}$$

We now calculate the effective diffusion coefficient. Notice that D_{eff} from Eq. (22) can be rewritten (after some integrations by parts) as follows:

$$\beta D_{\mathrm{eff}} = J_\infty L\beta \left(\int_{-L/2}^{L/2} [S(x) - x] P_\infty(x) dx - \frac{1}{L} \int_{-L/2}^{L/2} [S(x) - x] dx \right) + L \int_{-L/2}^{L/2} Q_\infty(x) P_\infty(x) dx, \tag{E22}$$

where we have chosen, without loss of generality, the integration region from $-L/2$ to $L/2$.

The integrals involved in the above equation can again be evaluated using the steepest descent method. Since the major contribution comes from points near x^*, we can use the asymptotic expressions (E10), (E18), and (E20). Substituting these expressions in Eq. (E22), applying the change of variable $t = (\mu\beta)^{1/3}x$, and extending the integration limits to $\pm\infty$ [17], we obtain

$$\beta D_{\mathrm{eff}} = L \frac{(\beta\mu)^{2/3}}{G_0} \left[\frac{L\Gamma_2}{G_0} - \frac{(\mu\beta)^{-1/3}\Gamma_3}{G_0} - \left(LT(-\infty) + \frac{L}{2} - \frac{(\mu\beta)^{-1/3}\Gamma_1}{G_1} \right) \right] + L \frac{(\mu\beta)^{1/3}}{G_1 G_0} \Gamma_4, \tag{E23}$$

which, ordered in powers of β, is

$$\beta D_{\mathrm{eff}} = \frac{L^2(\beta\mu)^{2/3}}{G_0} \left[\frac{1}{G_0}\Gamma_2 - \left(T(-\infty) + \frac{1}{2} \right) \right] + \frac{L(\beta\mu)^{1/3}}{G_0} \left(\frac{\Gamma_4}{G_1} - \frac{\Gamma_3}{G_0} + \frac{\Gamma_1}{G_1} \right). \tag{E24}$$

Here Γ_1, Γ_2, Γ_3, and Γ_4 are constants given in terms of quadratures,

$$\Gamma_1 := \int_{-\infty}^{\infty} tK_1(t) dt, \quad \Gamma_2 := \int_{-\infty}^{\infty} K_0(t) T(t) dt,$$

$$\Gamma_3 := \int_{-\infty}^{\infty} tK_0(t) dt, \quad \Gamma_4 := \int_{-\infty}^{\infty} K_0(t) K_1(t) dt.$$

Equation (E24) shows that the leading order in β as it goes to infinity is 2/3. Taking away the terms that go as $\beta^{1/3}$, we obtain

$$\beta D_{\mathrm{eff}} = \frac{L^2(\beta\mu)^{2/3}}{G_0} \left[\frac{1}{G_0}\Gamma_2 - \left(T(-\infty) + \frac{1}{2} \right) \right]. \tag{E25}$$

Finally, after evaluating numerically the integrals involved in the above expression we obtain Eq. (29) for the effective diffusion coefficient.

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[16] Strictly speaking, this is not an eigenfunction expansion, since the set $\{P_\infty(x)e^{iqS(x)}|q=2\pi m/L\ m\in\mathbb{Z}\}$ is not the set of eigenfunctions of the Fokker-Planck operator.
[17] This can be done since the integration limits become $\pm(\mu\beta)^{1/3}L/2$ instead of $\pm L/2$. We should also note that $(\mu\beta)^{1/3}L/2\approx(\beta F_c L/2)^{1/3}\gg 1$ since μ is approximately $4F_c/L^2$ (see Ref. [12]). This is a consequence of the fact that $V(x)\approx-\mu x^3$ is approximately valid in the whole interval $[-L/2,L/2]$.